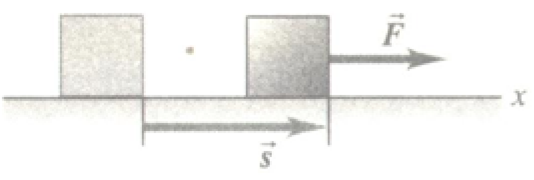
7.1
Work

When you push a car, you do work. You do work by exerting a *force* on a body while that body *moves* from one place to another, that is, undergoes a *displacement*. You do more work if the force is greater (you push harder on the car) or if the displacement is greater (push the car farther).

Consider a body that undergoes a displacement of magnitude *s* along a straight line. While the body moves, a constant force **F** acts on it in the same direction as the displacement **s**.



We define the ***work*** W done by this constant force as the product of the force magnitude *F* and the displacement magnitude *s*:

$$W = Fs \qquad (1)$$

The work done on the body is greater if either the force *F* or the displacement *s* is greater.

The SI unit of work is the ***joule*** (abbreviated as J). From the above equation we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. Therefore, in SI units

1 joule = (1 Newton) (1 meter)

$1\,\mathrm{J} = 1\,\mathrm{N \cdot m}$

In the British System the unit of force is the pound (lb), the unit of distance is the foot, and the unit of work is the *foot-pound* (ft.lb).

$1\,\mathrm{J} = 0.7376\,\mathrm{ft \cdot lb} \qquad 1\,\mathrm{ft \cdot lb} = 1.356\,\mathrm{J}$

If you push the car through a displacement s with a constant force **F** in the direction of motion, the amount of work you do on the car is given by $W = Fs$. If you push the car at an angle ϕ with the car's displacement, only the component of the force in the direction of the car's motion would be effective in moving the car.

When the force **F** and the displacement s have different directions, we take the component of **F** in the direction of the displacement s, and we define the work as the product of this component and the magnitude of the displacement. The component of **F** in the direction of **s** is $F\cos\phi$, so

$$W = Fs\cos\phi \qquad (2)$$

We are assuming that F and ϕ are constant during the displacement. If ϕ = 0, so that **F** and **s** are in the same direction, then $\cos\phi = 1$ and we get back the equation (1).

Equation (2) has the form of the scalar product of two vectors $\mathbf{A} \cdot \mathbf{B} = AB\cos\phi$. Using this, we can write Eq. (2) as

$$W = \mathbf{F} \cdot \mathbf{s} \qquad (3)$$

Work is a scalar quantity, even though it is calculated by using two vector quantities (force and displacement). A 5-N force toward the east acting on a body that moves 6 m to the east does exactly the same work as a 5-N force toward the north acting on a body that moves 6 m to the north.

Example

You exert a steady force of magnitude 210 N on the stalled car as you push it a distance of 18 m. The car also has a flat tire, so to make the car track straight you must push at an angle of 30° to the direction of motion. How much work you do? Suppose you push a second stalled car with a steady force of $\mathbf{F} = 160\mathbf{i} - 40\mathbf{j}$. The displacement of the car is $\mathbf{s} = 14\mathbf{i} + 11\mathbf{j}$. How much work you do in this case?

Solution

In each case the force is constant and displacement is along a straight line.

Case1:

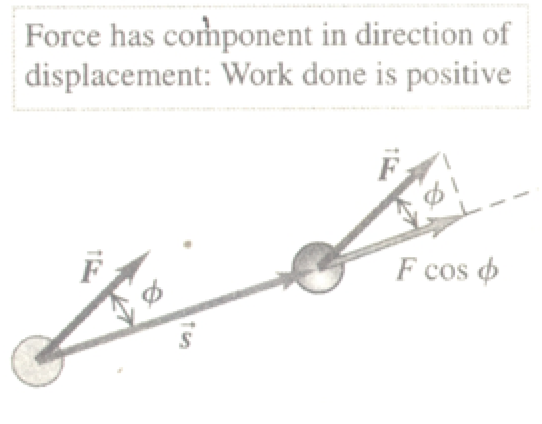
$$W = Fs\cos\phi = 210 \times 18 \times \cos 30^\circ = 3.3 \times 10^3\,\mathrm{J}$$

Case 2:

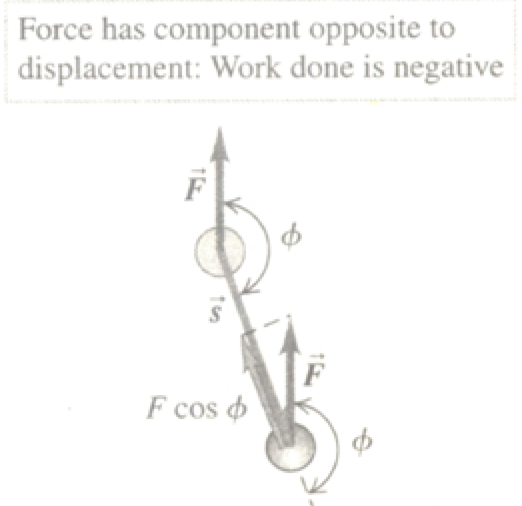
The components of **F** are $F_x = 160\,\mathrm{N}$ and $F_y = -40\,\mathrm{N}$. The components of **s** are $x = 14\,\mathrm{m}$ and $y = 11\,\mathrm{m}$. Hence,

$$\mathbf{W} = \mathbf{F}.\mathbf{s} = F_x x + F_y y = 160 \times 14 + (-40) \times 11 = 1.8 \times 10^3\,\mathrm{J}$$

In the example above, the work done in pushing the cars was positive. Work can also be negative or zero. When the force has a component in the *same direction* as the displacement (ϕ between zero and 90°), $\cos\phi$ is positive and the work is *positive*.



When the force has a component *opposite* to the displacement (ϕ between 90° and 180°), $\cos\phi$ is negative and the work is *negative*.



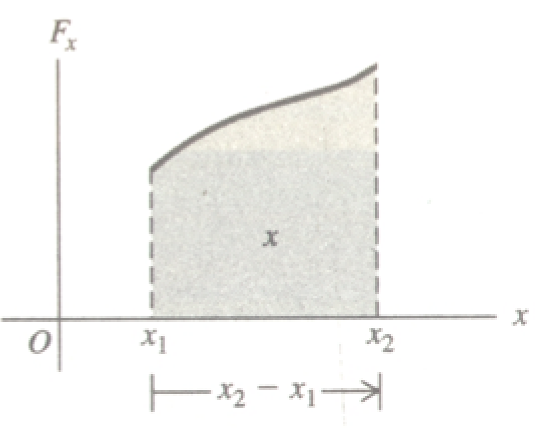
When the force is *perpendicular* to the displacement, ϕ = 90° and the work done by the force is *zero*.

7.2

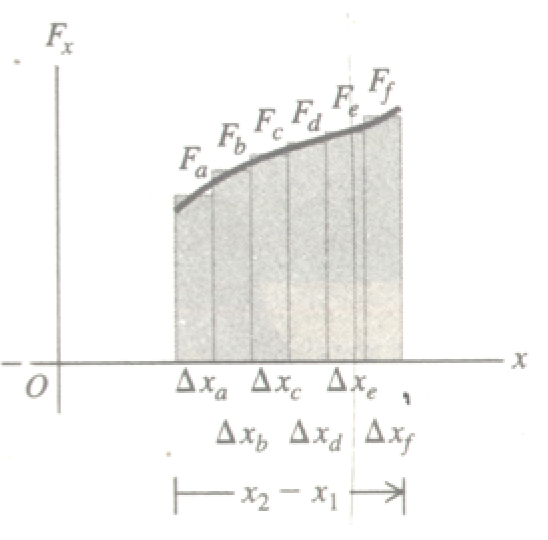
Work of a Variable Force

We consider straight-line motion with a force that is directed along the line but with an x-component F_x that may change as the body moves. For example, imagine a train moving on a straight track with the throttle setting being constantly changed.

Suppose a particle moves along the *x*-axis from point x_1 to x_2. The figure below is a graph of the *x*-component of the force as a function of the particle's coordinate *x*.



To find the work done by this force, we divide the total displacement into small segments $\Delta x_a, \Delta x_b$, and so on.



We approximate the work done by the force during the segment Δx_a as the average force F_a in that segment multiplied by the displacement Δx_a. We do this for each segment and then add the results for all the segments. The work done by the force in the total displacement from x_1 to x_2 is approximately

$$W = F_a \Delta x_a + F_b \Delta x_b + \cdots$$

As the number of segments becomes very large and the width of each becomes very small, this sum becomes (in the limit) the *integral* of F_x from x_1 to x_2:

$$W = \int_{x_1}^{x_2} F_x dx$$

We note that $F_a \Delta x_a$ represents the *area* of the first vertical strip and that the integral represents the area under the curve between x_1 and x_2.

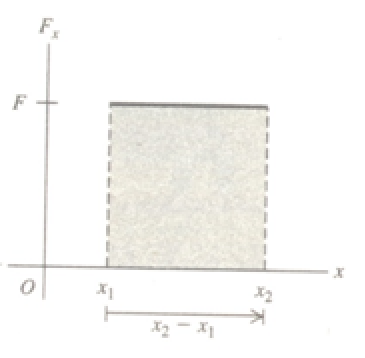
On a graph of force as a function of position, the total work done by the force is represented by the area under the curve between the initial and final positions.

An alternative interpretation is that the work *W* equals the average force that acts over the entire displacement, multiplied by the displacement.

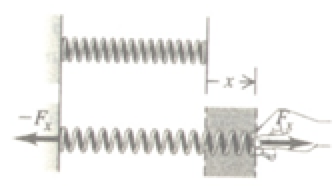
The above equation also applies if F_x, the *x*-component of force, is constant. In that case, F_x may be taken outside the integral:

$$W = \int_{x_1}^{x_2} F_x dx = F_x \int_{x_1}^{x_2} dx = F_x(x_2 - x_1) = Fs$$

The interpretation of work as the area under the curve of F_x as a function of *x* also holds for a constant force; $W = Fs$ is the area of a rectangle of height *F* and width *s*.



We consider a stretched spring. To keep a spring stretched beyond its unstretched length by an amount *x*, we have to apply a force with magnitude *F* at each end.

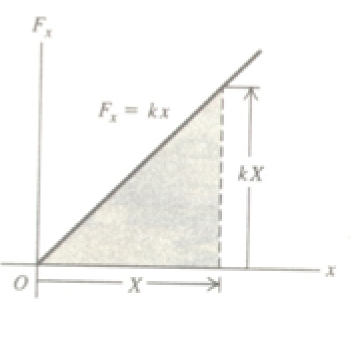


If the elongation *x* is not too great, we find that the force we apply to the right-hand end has an *x*-component directly proportional to *x*:

$$F_x = kx$$

where *k* is a constant called the **force constant** (or spring constant) of the spring. The observation that elongation is directly proportional to force for elongations that are not too great was made by Robert Hooke in 1678 and is known as Hooke's law.

To stretch a spring, we must do work. We apply equal and opposite forces to the ends of the spring and gradually increase the forces. We hold the left end stationary, so the force we apply at this end does no work. The force at the moving end does do work. Figure below is a graph of F_x as a function of *x*, the elongation of the spring.



The work done by this force when the elongation goes from zero to a maximum value of *X* is

$$W = \int_0^X F_x dx = \int_0^X kx dx = \frac{1}{2}kX^2 \qquad \text{(a)}$$

We can also obtain this result graphically. The area of the shaded triangle in the figure above, representing the total work done by the force, is equal to half the product of the base and altitude,

$$W = \frac{1}{2}(X)(kX) = \frac{1}{2}kX^2$$

This equation also says that the work is the average force $kX/2$ multiplied by the total displacement X. We see that the total work is proportional to the square of the final elongation X. To stretch a spring by 2 cm, you must do four times as much work as is needed to stretch it by 1 cm.

The above equation assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 is

$$W = \int_{x_1}^{x_2} F_x dx = \int_{x_1}^{x_2} kx dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad \text{(b)}$$

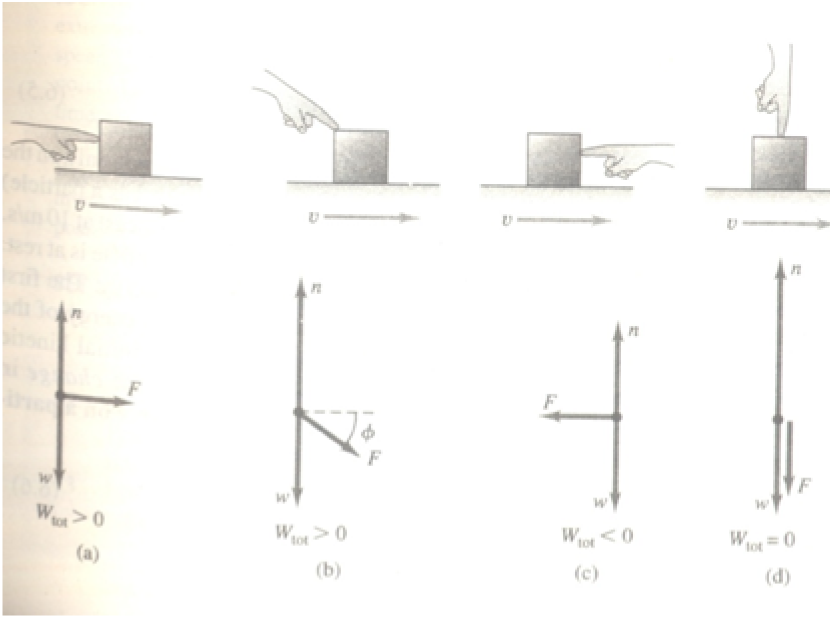
If the spring has spaces between the coils when it is unstretched, then it can also be compressed, and Hooke's law holds for compression as well as stretching. In this case the force *F* and the displacement *x* are in the opposite directions from those considered earlier, and so both F_x and x are negative. Since both F_x and x are reversed, the force again is in the same direction as the displacement, and the work done by F_x is again positive. So the total work is still given by equations (a) or (b), even when *X* is negative or either or both of x_1 and x_2 are negative.

We note that the work given by equation (b) is the work that you must do on a spring to change its length. For example, if you stretch a spring that's originally relaxed, then $x_1 = 0, x_2 > 0$, and $W > 0$. That's because the force you apply to one end of the spring is in the same direction as the displacement and the work you do is positive. By contrast, the work that the spring does on whatever it is attached to is given by the negative of equation (b). Thus, as you pull on the spring, the spring does negative work on you.

7.3

Kinetic Energy

The total work done on a body by external forces is related to the body's displacement – that is to changes in its position. The total work is also related to changes in the speed of the body as shown below.



The above is an example of a block sliding on a frictionless table. The forces acting on the block are its weight **w**, the normal force **n**, and the force **F** exerted on it by the hand. In Fig. a, the net force on the block is in the direction of its motion. From Newton's second law, this means that the block speeds up; this also means that the total work W_{tot} done on the block is positive. The total work is also positive in Fig. b. The block again speeds up. The component $F\cos\phi$ causes the acceleration and it contributes to W_{tot}. The total work is negative in Fig. c because the net force opposes the displacement; in this case the block slows down. The net force is zero in Fig. d, so the speed of the block stays the same and the total work done on the block is zero. We can conclude that when a particle undergoes a displacement, it speeds up if $W_{tot} > 0$, slows down if $W_{tot} < 0$, and maintains the same speed if $W_{tot} = 0$.

We generalize these observations. Consider a particle with mass *m* moving along the *x*-axis under the action of a constant net force with magnitude F directed along the positive *x*-axis. The particle's acceleration is constant. Suppose the speed changes from v_1 to v_2 while the particle undergoes a displacement $s = x_2 - x_1$ from point x_1 to x_2. Using constant-acceleration equation,

$$a_x = \frac{v_2^2 - v_1^2}{2s}$$

$$F = ma_x = m\frac{v_2^2 - v_1^2}{2s}$$

$$Fs = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all forces acting on the particle. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** K of the particle:

$$K = \frac{1}{2}mv^2$$

Like work, the kinetic energy of a particle is a scalar quantity; it depends only on the particle's mass and speed, not its direction of motion. A car, viewed as a particle, has the same kinetic energy when going north at 10 m/s as when going east at 10 m/s. Kinetic energy can never be negative, and it is zero only when the particle is at rest.

The first term on the right side of the equation $Fs = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$ is the final kinetic energy of the particle, while the second term is the initial kinetic energy. The difference between these terms is the change in kinetic energy. Therefore, ***the work done by the net force on a particle equals the change in the particle's kinetic energy***:

$$W_{tot} = K_2 - K_1 = \Delta K$$

This result is known as the **work-energy theorem**.

When W_{tot} is positive; K_2 is greater than K_1, the kinetic energy increases, and the particle is going faster at the end of the displacement than at the beginning. When W_{tot} is negative, the kinetic energy decreases, and the speed is less after the displacement. When $W_{tot} = 0$, the initial and final kinetic energies K_1 and K_2 are the same and the speed is unchanged. It is important to note that the work-energy theorem by itself tells us only about changes in speed, not velocity, since the kinetic energy carries no information about the direction of motion.

The kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy.

Because we used Newton's laws in deriving the work-energy theorem, we can use only in an inertial frame of reference. The speeds that we use to compute the kinetic energies and the distance that we use to compute work must be measured in an inertial frame.

We have derived the work-energy theorem for the special case of straight line motion with constant forces. In the next module, we will show that the theorem is valid in general, even when the forces are not constant and the particle's trajectory is curved.

7.5 Power

The definition of work makes no reference to time, and often we need to know how quickly the work is done. This can be described in terms of power. **Power** is the time rate at which work is done. Like work and energy, power is a scalar quantity.

When a quantity of work ΔW is done during a time interval Δt, the average work done per unit time or ***average power*** P_{av} is defined to be

$$P_{av} = \frac{\Delta W}{\Delta t}$$

We can define ***instantaneous power*** P as the limit of the quotient in the above equation as Δt approaches zero:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

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The SI unit of power is the **watt** (W). One watt equals one joule per second (1 W = 1 J/s). In the British system, work is expressed in foot-pounds, and the unit of power is the foot-pound per second. A larger unit called the horsepower (hp) is also used:

1 hp = 550 ft.lb/s = 33,000 ft.lb/min

That is, a 1-hp motor running at full load does 33,000 ft-lb of work every minute. A conversion factor is

1 hp = 746 W = 0.746 kW

That is, 1 horsepower equals about ¾ of a kilowatt.

The watt is a familiar unit of electrical power; a 100-W light bulb converts 100 J of electrical energy into light and heat each second.

The units of power can be used to define new units of work and energy. The *kilowatt-hour* (kWh) is the usual commercial unit of electrical energy. One kilowatt-hour is the total work done in 1 hour (3600 s) when the power is 1 kilowatt (10^3 J/s), so

1 kWh = (10^3 J/s) (3600 s) = $3.6\times10^6\ J$ = 3.6 MJ

The kilowatt-hour is a unit of work or energy, not power.

In mechanics, we can express power in terms of force and velocity. Suppose that a force **F** acts on a body while it undergoes a vector displacement Δs. If $F_\parallel$ is the component of **F** tangent to the path (parallel to Δs), then the work done by the force is $\Delta W = F_\parallel \Delta s$, and the average power is

$$P_{av} = F_\parallel \frac{\Delta s}{\Delta t} = F_\parallel v_{av}$$

Instantaneous power *P* is the limit of this expression as $\Delta t \to 0$:

$$P = F_\parallel v$$

where *v* is the magnitude of the instantaneous velocity. We can also express this equation in terms of the scalar product:

$$P = \mathbf{F}\cdot\mathbf{v}$$

Example

Each of the two jet engines on a Boeing 767 airliner develops a thrust (a forward force on the airplane) of 197,000 N. When the airplane is flying at 250 m/s, what horsepower does each engine develop?

Solution

The thrust is in the direction of motion, so $F_\parallel = 197{,}000\ \text{N}$. At $v = 250\ \text{m/s}$, each engine develops the power

$$P = F_\parallel v = 197{,}000 \times 250 = 4.93\times10^7\ \text{W}$$
$$= \left(4.93\times10^7\ \text{W}\right)\frac{1\,\text{hp}}{746\ \text{W}} = 66{,}000\ \text{hp}$$

Example

A runner with mass 50.0 kg runs up to the top of the 443-m-high hill. To lift himself to the top in 15.0 minutes, what must be his average power output in watts? In kilowatts? In horsepower?

Solution

We will treat the runner as a particle of mass *m*. Lifting a mass *m* against gravity requires an amount of work equal to the weight mg multiplied by the height *h* it is lifted. Hence the work he must do is

$$W = mgh = 50.0\times9.80\times443 = 2.17\times10^5\ \text{J}$$

The time is 15.0 min = 900 s, so the average power is

$$P_{av} = \frac{2.17\times10^5}{900} = 241\ \text{W} = 0.241\ \text{kW} = 0.323\ \text{hp}$$

In the alternative approach, we calculate the power:

The force exerted is vertical, and the average vertical component of velocity is 443/900=0.492 m/s, so the average power is

$$P_{av} = F_\parallel v_{av} = (mg)v_{av} = (50.0)(9.80)(0.492) = 241\ \text{W}$$

Example

When its 75-kW (100 hp) engine is generating full power, a small single-engine airplane with mass 700 kg gains altitude at a rate of 2.5 m/s. What fraction of the engine power is being used to make the airplane climb? (The remainder is used to overcome the effects of air resistance and of inefficiencies in the propeller and engine.)

Solution: The rate at which work is being done against gravity is

$$P = Fv = (700\times9.80)\times(2.5) = 17.15\ \text{kW}$$

This is the part of the engine power that is being used to make the airplane climb. This, as a fraction of the total, is

$$\text{fraction} = \frac{17.15\ \text{kW}}{75\ \text{kW}} = 0.22$$

Example

An elevator has a mass 600 kg, not including passengers. The elevator is designed to ascend, at constant speed, a vertical distance of 20.0 m (five floors) in 16.0 s, and it is driven by a motor that can provide up to 40 hp to the elevator. What is the maximum number of passengers that can ride in the elevator? Assume that an average passenger has mass 65.0 kg.

Solution: The power that is delivered to elevator is

$$P = (40\ \text{hp})\times\left(\frac{746\ \text{W}}{1\ \text{hp}}\right) = 2.984\times10^4\ \text{W}$$

If m denote the total mass lifted

$$P = \frac{mgh}{t}$$

or

$$m = \frac{Pt}{gh} = \frac{2.984\times10^4\times16.0}{9.80\times20.0} = 2436\ \text{kg}$$

This is the total mass of elevator plus passengers. The mass of the passengers is 2436 − 600 = 1836 kg. The number of passengers is

$$\frac{1836}{65.0} = 28.2$$

So 28 passengers can ride.

Example

A particle is accelerated from rest by a constant net force.

1. Show the instantaneous power supplied by the net force is ma^2t.
2. To triple the acceleration at any given time, by what factor must the power be increased?
3. At *t* = 5.0 s the power supplied by the net force is 36 W. What must the power be at *t* = 15.0 s to maintain constant acceleration?

Solution:

1. $F = ma$ $\qquad v = v_0 + at = at$

   The instantaneous power is

   $$P = Fv = ma\times at = ma^2t$$

2. P is proportional to a^2. Triple a implies the increase of P by a factor of 9.
3. We have the relation $\frac{P}{t} = ma^2 = \text{constant}$. Therefore,

$$P_2 = P_1\frac{t_2}{t_1}$$
$$= 36\times\frac{15.0}{5.0} = 108\ \text{W}$$